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adds, "The magic that fights for us, the Venus-eye that ensnares even our opponents and blinds them, is the magic of extremes, the allurement of daring to the uttermost."[92]

Itself an extreme utterance, we say—but it may be safer to let the future decide that! In this strange world, the unexpected, the undreamed of, sometimes happens.

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ON THE NOTION OF INFINITY

WHENEVER a man writes a book to be read, he makes certain tacit assumptions with regard to the reader. He assumes, for example, a mind which can attend to things and relations between them, a mind which can choose among things, and which understands before and after. It would be absurd to tell a man, in substance, that before he can understand "before," he must first understand something else, and that after he understands this, he may be able to understand "after"; that if he will choose to listen you will tell him the meaning of "choose," and that certain relations among things will make clear to him what "things" and "relations" are. Naturally, in writing this paper on the notion of infinity, we shall suppose that the reader possesses such a mind, and the assumption gives the key to our argument. We shall make *choices* among the natural numbers 1, 2, 3 . . . ; some choices will be made *before* and others *after*, and it will turn out that the notions *choice* and *before and after* are essential to the notion of infinity.

But we must be on our guard in one respect at the outset. Difficulty arises from the use of certain phrases by men who best understand the infinity-concept, and if these phrases are misunderstood, it may be claimed that they lead to contradictions. We hear the phrase "when x is infinite" or "when x goes to infinity" as if infinity were a place where x might go and stay. We hear that there are "definite infinite numbers," and we interpret this to mean "there are infinite numbers that can be fixed by counting." Of course, such statements are contradictory, and it is useless to consider the question, "What finite number is infinite?"

We must ask about any statement, not only, "Is it true?" but, "What does it mean?" for it may easily happen that a statement is true with one meaning, and false with another, so that the statements, "It is true," "It is false," are equally correct and equally careless. To guard against such difficulties, let us introduce a sort of vocabulary giving some of the terms that we shall use, together with their meanings for this paper.

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[92] "Will to Power," § 749. In "Ecce Homo," III., ix, § 2, he says, in speaking of the new hopes and tasks for mankind, "I am their happy messenger" (*cf.* IV., § 1).

We shall be concerned, first, with the *natural numbers,* 1, 2, 3, *et cetera.* These numbers are defined by the *counting process,* which is itself undefined except for the assumption that we may "count one more" than any number already defined. A number is said to be *greater* than another if it is defined by the counting process after the other; then the other is said to be *less* than the one.

Corresponding to the natural numbers we shall consider *sets* or *groups* of *elements.* The term group is undefined, but is used in the sense "a group of five chairs." In general, a group has *parts* or *subgroups* where the elements of the subgroup are some, but not all of the elements of the group; as, a group of five chairs has a subgroup of three chairs. Each group has associated with it a *cardinal number,* as the group of five chairs has associated with it the cardinal number, 5.

Groups are compared by the process of *one-to-one correspondence.* This term also is undefined, but is used in the sense of the old jingle

> "I saw five brave maids
> Sitting on five broad beds
> Braiding broad braids,"

where there is a one-to-one correspondence between maids, beds, and braids.

A group is *finite* if its cardinal number is one of the natural numbers, 1, 2, 3, *et cetera.*

A finite group is said to be *less* than a second group if the elements of the first may be paired one-to-one with the elements of a part of the second.

We assume that a group may be formed by "adding one element" to any existing finite group.

With this vocabulary at our disposal let us go on to define an *infinite group.* We may give Cantor's definition,[1] "a group is said to be infinite if there is no finite group with whose elements the elements of the group may be paired one-to-one," or what is the same thing[2] "a given group is said to be infinite when, if a is any finite number that has been chosen, the group has a subgroup of a elements."

The word *any* as used in this definition must be distinguished from *some.* If "so-and-so is true of any element of a group," we know so-and-so will hold with regard to a chosen element before we choose the element. If "so-and-so is true of some element of a group" we do not know whether so-and-so will hold with regard to a chosen ele-

[1] "G. Cantor, "Zur Begründung der transfiniten Mengenlehre." *Math. An.,* Vol. 46, page 492.

[2] As may be shown, although it is not shown here.

ment until after we have chosen the element. Speaking loosely, *any* element of a group is an element that has not yet been chosen; *some* (a certain) element of a group is an element that has been chosen.

The number *a* of our definition is both any number and some number, according to the point of view. First, a group is supposed to be fixed according to some chosen law. With respect to all that follows the group is *some* group, and with respect to the group, *a* is *any* number, that is, *a* has not yet been chosen. But there comes a time when *a* is supposed to be chosen. With respect to all that follows, namely, the choice of the subgroup, the number *a* is *some* number, that is, with respect to the subgroup *a* is a number that has been chosen.

Then our definition means that the infinite group is chosen before the number *a*, and the subgroup is chosen after the number *a*.

This peculiar order of choices must be preserved in a definition of infinite group, for

(1) A *finite* group will fulfil all the conditions if both the group and the subgroup may be chosen after the number *a*, as follows: first, choose *a*, say, 20; then a finite group, say, of 30 things; then a subgroup of 20 things. The finite group of 30 things fulfils the conditions. And evidently, it does not matter how large *a* is taken. We could fulfil the conditions in the same way if *a* were chosen equal to a million or even a larger number.

(2) There is *no* group that will fulfil the conditions if both the group and the subgroup must be chosen before the number *a*. Choose, say, the group of finite integers and a finite subgroup at pleasure, say, the first million integers. Now suppose an opponent chooses the number *a* equal to a million and one. There is nothing to prevent his doing so, but in that case the conditions of the definition are not fulfilled, even by the group of finite integers.

The order of procedure prescribed by our definition is as follows: First, let the group be given according to a certain law; for example, let the group be the group of finite integers 1, 2, 3, . . . With the group defined we give our opponent a perfectly free choice of the number *a*. The group is infinite in case it has a subgroup of *a* elements no matter what number *a* happened to be chosen. Evidently, the group of finite integers has such a subgroup corresponding to any chosen *a*, namely, the group of the first *a* integers; if *a* is a million, the first million integers.

If some one objects that we have been talking only about finite groups and have not defined an infinite group at all, we may challenge him to produce a finite group which may be treated in the same way, namely, first, let the group be defined; second, give an opponent a perfectly free choice of the number *a*, and third, find in the given group a subgroup of *a* elements.

Beside the definition already given, Cantor gives the equivalent definition "a group is infinite if its elements may be paired one-to-one with the elements of a part of the group." But one might say, "who ever saw such a group?" It surely does not belong to our sense experience. If the definition is to have value, such a group must be shown to *exist*. Cantor says that the group of natural numbers is such a group because its elements may be paired one-to-one with the elements of the group of even numbers as follows:

$$\begin{array}{l} 1, 2, 3, 4, 5 \ldots, \\ 2, 4, 6, 8, 10 \ldots. \end{array}$$

But no one has ever paired more than a finite number of the elements of these groups. Then Cantor's statement means, "if we choose any number we can find its double, and if we choose any even number, we can find its half," and we can perform this process (a finite number of times) as often as we like.

With a *finite* group, say, of 21 elements we may pair an element of a part with an element of the whole 20 times, if we wish, but not more than 20 times. On the other hand, with an infinite group we do not have to stop pairing an element of the part with an element of the whole unless we choose to do so, that is, in case of the infinite group, there is no finite number that corresponds to the number 20, above. And yet it is not exact to say that we can pair *all* the elements of the infinite group with their doubles, because all the elements of the infinite group can not be chosen.

To be sure, Cantor speaks of "the totality of all finite numbers," but he does not mean "all numbers that have been defined" or "are supposed to have been defined," but rather those that *may* be defined. We can not suppose that all the finite numbers have been fixed. What has been fixed is a law by which *may* be fixed as many finite numbers as we wish. Then, the difference between finite and infinite depends fundamentally on the difference between what has been done and what may be done. It demands for its understanding a mind such as we assumed at the outset, a mind that can choose among things with a sense of before and after in its choices.

This relation of before and after will serve also to interpret Cantor's proof that the set of real numbers between zero and one is non-denumerable.[3] The proof runs as follows: Suppose that a law could arrange the never-ending decimals[4] between zero and one in a denumerable order corresponding to the numbers 1, 2, 3, . . . as follows:

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[3] A denumerable set can be put into one-to-one correspondence with the set of natural numbers 1, 2, 3 .... In such a set, each element has a definite place with reference to the other elements, a third place or a fifth or tenth place. The set of even integers is denumerable because each number has its place, for example, the number 24 has the twelfth place.

[4] The digits are supposed to follow one another according to a fixed law,

$$\begin{array}{ll}
(1) & \cdot a_1\ a_2\ a_3\ a_4\ \cdots, \\
(2) & \cdot b_1\ b_2\ b_3\ b_4\ \cdots, \\
(3) & \cdot c_1\ c_2\ c_3\ c_4\ \cdots, \\
(4) & \cdot d_1\ d_2\ d_3\ d_4\ \cdots, \\
\multicolumn{2}{l}{\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

where the letters stand for digits, for example, $\cdot a_1 a_2 a_3 a_4 \cdots$ might be $\cdot 1\,5\,8\,7 \cdots$. Then there is at least one never-ending decimal unaccounted for in the array, namely, $\cdot a'_1 b'_2 c'_3 d'_4 \cdots$, where a'_1 is a digit that is *not* a_1, b'_2 is not b_2, *et cetera,* because this decimal differs from any in the array in at least one place; it differs from the first decimal in the first place, from the second decimal in the second place, and so on.

But, the objection is raised, "Does not this exception number occur lower down in the array?" If it occurs in the array it must have a definite place corresponding to one of the natural numbers 1, 2, 3, . . . Suppose that this place is the nth. n is a fixed number, since the denumerable order is supposed *to have been chosen.* Since the decimals are never ending, the nth decimal has an nth digit, and this digit can not belong both to the nth decimal and the exception number by the law of formation of the latter. Therefore, if we *fix* any denumerable order of never-ending decimals, we can *then* find a never-ending decimal which has no place in the order. In fact, the very writing of a number that has a place in the order helps to create a number that has no place in the order; and just so long as there are numbers with definite places, just so long is there at least one number without definite place. According to the conditions, the order has been chosen "before" and the exception number may be chosen "after," and the theorem is proved.

The rôle played in modern analysis by this concept of before and after may be illustrated further from the theory of limits. Perhaps the simplest illustration is the theorem that the limit, as x becomes infinite, of the fraction $1/x$ is zero. The theorem *means* (and here lies the difficulty) "if a constant e has been chosen, no matter how small, then there is a constant x_1, such that when x is greater than x_1, $1/x$ is less than e." The theorem is *true,* because, *after e has been chosen,* we may choose $x_1 = 1/e$; and if x is greater than x_1, then $1/x$ is less than e. In other words, the larger the denominator the smaller the fraction, and we may take the denominator of the fraction $1/x$ large enough to make the value of the fraction as small as we please. If x is greater than 10, $1/x$ is less than 1/10; if x is

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and the phrase "never-ending decimal" means strictly the limit of a decimal of n digits as n becomes infinite.

greater than 100, $1/x$ is less than $1/100$; if x is greater than $1/e$, $1/x$ is less than e.

An imaginary opponent objects, "Of course, you can do all this after e has been chosen, but suppose e had not been chosen first. I might have chosen e differently." We reply, "If e had not been chosen first, then our statement would not have the meaning which we have assigned to it, for it is precisely in this choosing first or second that the essential of the concept lies. You had a perfectly free choice of e. Why didn't you make the most of it?"

The idea of before and after enters also into the notions of function, continuity, uniform continuity, convergence, uniform convergence, *et cetera,* of which we may not treat. It is evidently the idea of *time,* not in the sense of measured time, nor in Bergson's sense of "duration," but in the sense that one thing is done or supposed done before another, as one move is made before another in a game of chess. If the choices of numbers are actually made, the time in question is real; if the choices are only supposed to be made, the time is fictitious as a dream time is fictitious, but as the *idea* of time is necessary to the meaning of a dream, so the idea of time is necessary to the concepts of mathematics with which we have been occupied.

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REVIEWS AND ABSTRACTS OF LITERATURE

Henri Bergson: A Study in Radical Evolution. EMIL CARL WILM. New York: Sturgis and Walton Company. 1914. Pp. xv + 193.

Henri Bergson: An Account of His Life and Philosophy. ALGOT RUHE and NANCY MARGARET PAUL. London: Macmillan and Company. 1914. Pp. vii + 245.

The disposition toward Bergsonism in these two books is a contrast, and this enhances their interest, though neither lacks interest on its own merits. They are instructive studies, and more: each is literature of superior elegance and charm. Otherwise they are as different from each other as possible.

The book by Professor Wilm is the lighter in every way, of little more than half the length of the other, and composed in a style exceedingly easy to read. Rather than the study or class-room, I should say that the congenial environment for its reading is a hammock. Intention to deal with "the subtler complications" of Bergson's thought is explicitly renounced in the Preface; and of the classes of people whom Bergsonism interests, "the reading laity" and "scientists and professional students of philosophy," the book is addressed to the first. The Preface, taken by itself, suggests, what the succeeding text does not, that the author regards Bergson's work very lightly—as the *dernier cri* from Paris,